Closed form representations and properties of the generalised Wendland functions

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Abstract

In this paper we investigate the generalisation of Wendland's compactly supported radial basis functions to the case where the smoothness parameter is not assumed to be a positive integer or half-integer and the ℓ parameter need not take on the minimal value for positive-definiteness. We derive sufficient and necessary conditions for the generalised Wendland functions to be positive definite and deduce the native spaces that they generate. We also provide closed form representations for the generalised Wendland functions in the case when the smoothness parameter is an integer (for any value of ℓ) in any dimension, as well as closed form representations for the Fourier transform when the smoothness parameter is a positive integer or half-integer.

1 Generalised Wendland Functions

Positive definite functions are frequently found at the heart of scattered data fitting algorithms both in Euclidean space and on spheres: see [16]. The aim of this paper is to investigate a large class of such functions. The following definition fixes the notation for what follows.

Definition 1.1. *A function $\phi : [0, \infty) \to \mathbb{R}$ is said to generate a strictly positive definite radial function on $\mathbb{R}^d$, if, for any $n > 2$ distinct locations $x_1, \ldots, x_n \in \mathbb{R}^d$, the following $n \times n$ distance matrix*

$$\Big(\phi(\|x_j - x_k\|)\Big)_{j,k=1}^{n}, \tag{1.1}$$

where $\|\cdot\|$ denotes the Euclidean norm, is positive definite.

For such functions we have the following characterization theorem (see [6] p34).

Theorem 1.2. *A continuous function $\phi : [0, \infty) \to \mathbb{R}$ such that $r \mapsto r^{d-1}\phi(r) \in L_1[0, \infty)$ generates a strictly positive definite radial function on*

$\mathbb{R}^d$ if and only if the $d-$dimensional Fourier transform

$$\mathcal{F}_d\phi(z) = z^{1-\frac{d}{2}} \int_0^\infty \phi(y)\, y^{\frac{d}{2}}\, J_{\frac{d}{2}-1}(yz)\, dy, \tag{1.2}$$

(where $J_\nu(\cdot)$ denotes the Bessel function of the first kind with order ν) is non-negative and not identically equal to zero.

In this paper we will investigate the family of parameterised basis functions defined by:

$$\phi_{\mu,\alpha}(r) := \frac{1}{2^{\alpha-1}\Gamma(\alpha)} \int_r^1 (1-t)^\mu\, t\, \left(t^2 - r^2\right)^{\alpha-1} \mathrm{d}t \quad \text{for} \quad r \in [0,1], \tag{1.3}$$

where $\mu > -1$, $\alpha > 0$ and $\Gamma(\cdot)$ denotes the Gamma function

Before we embark on our investigation we briefly review what is already known of this family. Firstly, it is well known (see [3]) that if $\alpha = k \in \{0, 1, 2, \ldots\}$ then the function $\phi_{\mu,k}$ generates a strictly positive definite function on $\mathbb{R}^d$ if and only if

$$\mu \geq \frac{d+1}{2} + k. \tag{1.4}$$

In particular, quoting [3], the function $\phi_{\mu,k}$ is $2k$ times differentiable at zero, positive, strictly decreasing on its support and has the form

$$\phi_{\mu,k}(r) = p_k(r)(1-r)_+^{\mu+k}, \tag{1.5}$$

where p_k is a polynomial of degree k with coefficients in μ and $(x)_+ := \max(x, 0)$. In [15] Wendland considers the case where

$$\mu = \ell := \left\lfloor \frac{d}{2} \right\rfloor + k + 1 \tag{1.6}$$

i.e., the smallest allowable integer that still allows positive definiteness. In this setting we can deduce from (1.5) that $\phi_{\ell,k}$ is a polynomial of degree $2k + \ell$ on the unit interval. Furthermore, it can be shown (see Chapter 10 of [16]) that, when d is odd, the function

$$\Phi(\mathbf{x}, \mathbf{y}) = \phi_{\frac{d+1}{2}+k,k}(\|\mathbf{x} - \mathbf{y}\|) \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d, \tag{1.7}$$

is the reproducing kernel of a Hilbert space which is norm equivalent to the integer order Sobolev space $H^{\frac{d+1}{2}+k}(\mathbb{R}^d)$.

In a relatively recent development Schaback [14] considered the case where α is taken to be a positive half-integer, i.e., where $\alpha = k + 1/2$. In this setting the following result is established (see [14] Theorem 3.1).

Thereom 1.3. *Let d be a fixed spacial dimension and k be a positive integer. Then if*

$$\mu \geq \left\lfloor \frac{d+1}{2} \right\rfloor + k + 1$$

then $\phi_{\mu,k+\frac{1}{2}}$ generates a positive definite function on $\mathbb{R}^d$.

Following Wendland's approach, we can devote particular attention to the case where $\mu = \ell = \lfloor \frac{d+1}{2} \rfloor + k + 1$, the smallest allowable integer that still allows positive definiteness; Schaback describes the resulting family $\phi_{\ell,k+\frac{1}{2}}$ as the missing Wendland functions and demonstrates (in analogy to the orignal Wendland functions) that, when d is even, the function

$$\Phi(\mathbf{x}, \mathbf{y}) = \phi_{\frac{d}{2}+k+1,k+\frac{1}{2}}(\|\mathbf{x} - \mathbf{y}\|) \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d, \tag{1.8}$$

is the reproducing kernel of a Hilbert space which is norm equivalent to the integer order Sobolev space $H^{\frac{d}{2}+k+1}(\mathbb{R}^d)$. An important distinction between the original Wendland functions and the missing Wendland functions is that the missing Wendland functions, whilst still being compactly supported, now have logarithmic and square-root multipliers of polynomial components.

In this article we present a more general investigation into the family of functions (1.3) which, building upon extant knowledge, we shall describe as the generalised Wendland functions. Specifically, for a given dimension d, we determine the full range of parameters μ and α for which the function $\phi_{\mu,\alpha}$ generates a d-dimensional positive definite function and, in addition, by examining the decay rate of the Fourier transforms of such functions, we also establish the nature of the reproducing Hilbert space. We also present closed form representations for $\phi_{\mu,k}$ $k \in \mathbb{N}_0$ and closed form representations for the Fourier transform of the original and missing Wendland functions.

2 The functions $\phi_{\mu,\alpha}$ and their Fourier transforms

In order to examine the range of parameters μ, α for which $\phi_{\mu,\alpha}$ generates a positive definite function on $\mathbb{R}^d$, we compute its $d-$dimensional Fourier transform. Using (1.2) and (1.3) we have

$$\mathcal{F}_d\phi_{\mu,\alpha}(z) = z^{1-\frac{d}{2}}\int_0^1 \phi_{\mu,\alpha}(y)y^{\frac{d}{2}}J_{\frac{d}{2}-1}(yz)\,\mathrm{d}y$$

$$= \frac{z^{1-\frac{d}{2}}}{2^{\alpha-1}\Gamma(\alpha)}\int_0^1\int_y^1 (1-t)^{\mu}t(t^2-y^2)^{\alpha-1}y^{\frac{d}{2}}J_{\frac{d}{2}-1}(yz)\,\mathrm{d}y\,\mathrm{d}t.$$

To develop this integral further we make the change of variables $s = y/t, x = t(0 \le y \le 1, y \le t \le 1)$ to yield

$$\mathcal{F}_d\phi_{\mu,\alpha}(z) = \frac{z^{1-\frac{d}{2}}}{2^{\alpha-1}\Gamma(\alpha)}\int_0^1\int_0^1 x^{2\alpha+\frac{d}{2}}(1-x)^{\mu}s^{\frac{d}{2}}(1-s^2)^{\alpha-1}J_{\frac{d}{2}-1}(zsx)\,\mathrm{d}s\,\mathrm{d}x.$$

From [5, 6.567.1], we have that

$$\int_0^1 s^{\nu+1}\left(1-s^2\right)^{\mu}J_{\nu}(bs)\,\mathrm{d}s = 2^{\mu}\Gamma(\mu+1)b^{-(\mu+1)}J_{\nu+\mu+1}(b), b > 0,$$

and so we can simplify the above expression to

$$\mathcal{F}_d\phi_{\mu,\alpha}(z) = z^{1-\frac{d}{2}-\alpha}\int_0^1 x^{\alpha+\frac{d}{2}}(1-x)^{\mu}J_{\alpha+\frac{d}{2}-1}(zx)\,\mathrm{d}x. \tag{2.1}$$

The following identity is taken from [5, 6.569]

$$\int_0^1 x^{\lambda}(1-x)^{\mu-1}J_{\nu}(ax)\,\mathrm{d}x = \frac{\Gamma(\mu)\Gamma(\lambda+\nu+1)2^{-\nu}a^{\nu}}{\Gamma(\nu+1)\Gamma(\lambda+\mu+\nu+1)}$$
$$\times\, {}_2F_3\left(\frac{\lambda+\nu+1}{2},\frac{\lambda+\nu+2}{2};\nu+1,\frac{\lambda+\nu+\mu+1}{2},\frac{\lambda+\nu+\mu+1}{2};-\frac{z^2}{4}\right),$$

where ${}_2F_3(a_1,a_2;b_1,b_2,b_3;z)$ denotes the hypergeometric function (see [1], 15.1.1) defined by

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) := \sum_{j=0}^{\infty}\frac{\Pi_{i=1}^{p}(a_i)_j}{\Pi_{i=1}^{q}(b_i)_j}\frac{z^j}{j!}, \tag{2.2}$$

with $p=2$ and $q=3$ and where

$$(c)_n := c(c+1)\cdots(c+n-1) = \frac{\Gamma(c+n)}{\Gamma(c)}, \quad n \ge 1 \tag{2.3}$$

denotes the Pochhammer symbol, with $(c)_0 = 1$.

Applying this identity allows us to conclude that

$$\mathcal{F}_d\phi_{\mu,\alpha}(z) = \frac{\Gamma(\mu+1)\Gamma(2\alpha+d)}{\Gamma(\alpha+\frac{d}{2})2^{\alpha+\frac{d}{2}-1}\Gamma(2\alpha+d+\mu+1)}$$
$$\times\, {}_2F_3\left(\frac{d}{2}+\alpha, \frac{d+1}{2}+\alpha; \frac{d}{2}+\alpha\frac{\mu+d+1}{2}+\alpha, \frac{\mu+d+2}{2}+\alpha; -\frac{z^2}{4}\right).$$

We notice that, in the terminology of the hypergeometric functions, we have, in the above example, the case where $a_1 = b_1$ and thus, by (2.2) this ${}_2F_3$ function collapses to a ${}_1F_2$ function. With this observation and the preceding development we have established the following theorem

Theorem 2.1. *The d-dimensional Fourier transform of the generalised Wendland functions $\phi_{\mu,\alpha}$, is given by*

$$\mathcal{F}_d\phi_{\mu,\alpha}(z) = C_d^{\mu,\alpha}\, {}_1F_2\left(\frac{d+1}{2}+\alpha; \frac{\mu+d+1}{2}+\alpha, \frac{\mu+d+2}{2}+\alpha; -\frac{z^2}{4}\right), \quad z > 0.$$

where

$$C_d^{\mu,\alpha} := \frac{\Gamma(\mu+1)\Gamma(2\alpha+d)}{2^{\alpha+\frac{d}{2}-1}\Gamma(\alpha+\frac{d}{2})\Gamma(2\alpha+d+\mu+1)} \tag{2.4}$$

The following result provides us with the range of parameters μ and α for which the function $\phi_{\mu,\alpha}$ generates a positive definite function on $\mathbb{R}^d$.

Theorem 2.2. *The generalised Wendland function $\phi_{\mu,\alpha}$ generates a positive definite function on $\mathbb{R}^d$ if and only if its parameters satisfy*

$$\mu \geq \frac{d+1}{2} + \alpha.$$

Proof. This follows directly from [10] which proves that

$${}_1F_2\left(a; a+\frac{b}{2}, a+\frac{b+1}{2}; -\frac{z^2}{4}\right) > 0, \quad z > 0,$$

for $b \geq 2a \geq 0$, for $b \geq a \geq 1$, or for $0 \leq a \leq 1, b \geq 1$. It is also proven that this function cannot be strictly positive for $0 \leq b < a$ or $a = b, 0 < a < 1$.

In our case, $a = \frac{d+1}{2} + \alpha > 1$ since $d \geq 1$ and $\alpha > 0$ and hence a sufficient and necessary condition reduces to $b \geq a$ which means that

$$\mu \geq \frac{d+1}{2} + \alpha.$$

$\square$

Now that we have established the correct parameter range for positive definiteness we turn next to examining the associated Hilbert function space $\mathcal{N}_{\phi_{\mu,\alpha}}$ whose reproducing kernel is the induced kernel

$$\Phi_{\mu,\alpha}(\mathbf{x},\mathbf{y}) = \phi_{\mu,\alpha}(\|\mathbf{x}-\mathbf{y}\|_2) \quad \mathbf{x},\mathbf{y} \in \mathbb{R}^d.$$

In order to establish such results we follow Wendland's approach and develop tight bounds upon the decay rate of the Fourier transform of the appropriate basis functions.

Theorem 2.3. *The d-dimensional Fourier transform of the generalised Wendland functions, $\mathcal{F}_d\phi_{\mu,\alpha}$, with $\mu \geq \alpha + \frac{d+1}{2}$, satisfies*

$$\mathcal{F}_d\phi_{\mu,\alpha}(z) = \Theta\left(z^{-(d+2\alpha+1)}\right) \quad \textit{for} \quad z \to \infty.$$

Proof. We need to show that for $z \geq z_0$, there exist two positive constants, c_1 and c_2 such that

$$c_1 \leq z^{d+2\alpha+1}\mathcal{F}_d\phi_{\mu,\alpha}(z) \leq c_2. \tag{2.5}$$

Using [7, 8], we have the following asymptotic expansion for $\mathcal{F}_d\phi_{\mu,\alpha}(z)$ as $z \to \infty$ and $|\arg(z)| < \frac{\pi}{2}$

$$\mathcal{F}_d\phi_{\mu,\alpha}(z) = \frac{\Gamma(\mu+d+1+2\alpha)}{\Gamma(\mu)} z^{-d-2\alpha-1}\left\{1+O(z^{-2})\right\}$$
$$+\frac{\Gamma(\mu+d+1+2\alpha)}{\Gamma\left(\frac{d+1}{2}+\alpha\right)} \frac{z^{-\left(\mu+\alpha+\frac{d+1}{2}\right)}}{2^{\left(\frac{d+1}{2}+\alpha\right)-1}} \left\{\cos\left[z-\frac{\pi}{2}\left(\mu+\alpha+\frac{d+1}{2}\right)\right]+O(z^{-1})\right\}.$$

Collecting terms not depending on z into constants c_3, c_4 and c_5 gives the following expression

$$z^{d+2\alpha+1}\mathcal{F}_d\phi_{\mu,\alpha}(z) = c_3\left\{1+O(z^{-2})\right\} + c_4 z^{\alpha+\frac{d+1}{2}-\mu}\left\{\cos(z-c_5)+O(z^{-1})\right\}. \tag{2.6}$$

Then for the upper bound, since $\cos(z)$ is bounded by 1 in absolute value, we can see that for $z \geq z_2$, there exists an $\epsilon_2 > 0$ such that

$$\begin{aligned} z^{d+2\alpha+1}\mathcal{F}_d\phi_{\mu,\alpha}(z) &\leq \left(c_3 + c_4 z^{\alpha+\frac{d+1}{2}-\mu}\right)(1+\epsilon_2) \\ &\leq (c_3+c_4)(1+\epsilon_2) \\ &=: c_2, \end{aligned}$$

which is positive since all its components are also positive. We proceed similarly for the lower bound and we first consider the case where $\mu = \frac{d+1}{2} + \alpha$. For $z \geq z_1$, there exists an $\epsilon_1 > 0$ such that

$$\begin{aligned} z^{d+2\alpha+1}\mathcal{F}_d\phi_{\mu,\alpha}(z) &\geq c_3(1-\epsilon_1) - c_4(1+\epsilon_1) \\ &= c_3 - c_4 - \epsilon_1(c_3+c_4) \\ &=: c_1. \end{aligned}$$

For small enough ϵ_1, $c_1 > 0$ since

$$\begin{aligned} c_3 - c_4 &= \Gamma(\mu+d+2\alpha+1)\left\{\frac{1}{\Gamma\left(\frac{d+1}{2}+\alpha\right)} - \frac{1}{\Gamma\left(\frac{d+1}{2}+\alpha\right)2^{\left(\frac{d+1}{2}+\alpha\right)-1}}\right\} \\ &> 0. \end{aligned}$$

Since the second term on the right hand side of (2.6) is decaying for $\mu > \frac{d+1}{2} + \alpha$, the existence of a lower bound in this case follows similarly. Setting $z_0 := \max(z_1, z_2)$ completes the proof. $\square$

With Theorem 2.3 established, we can appeal to the theory of radial basis functions (see [16]) to deduce the following.

Corollary 2.4. *Let $d \geq 1$ denote a fixed spatial dimension and $\alpha, \beta > 0$. The generalised Wendland function $\phi_{\frac{d+1}{2}+\alpha+\beta,\alpha}$ is reproducing in a Hilbert space which is isomorphic to the Sobolev space $H^{\frac{d+1}{2}+\alpha}\left(\mathbb{R}^d\right)$.*

2.1 Fourier transform dimension drop

With Theorem 2.1, we can deduce that

$$\mathcal{F}_d\phi_{\ell,k}(z) = C_d^{\ell,k}\,{}_1F_2\left(\frac{d+1}{2}+k;\frac{\ell+d+1}{2}+k,\frac{\ell+d+2}{2}+k;-\frac{z^2}{4}\right)$$

$$= C_d^{\ell,k}\,\underbrace{{}_1F_2\left(\frac{d}{2}+k+\frac{1}{2};\frac{\ell+d}{2}+k+\frac{1}{2},\frac{\ell+d+1}{2}+k+\frac{1}{2};-\frac{z^2}{4}\right)}_{=\mathcal{F}_{d-1}\phi_{\ell,k+\frac{1}{2}}(z)/C_{d-1}^{\ell,k+\frac{1}{2}}\text{ by Theorem 2.1}}$$

$$= \frac{C_d^{\ell,k}}{C_{d-1}^{\ell,k+\frac{1}{2}}}\mathcal{F}_{d-1}\phi_{\ell,k+\frac{1}{2}}(z)$$

Using (2.4) we see that $C_d^{\ell,k} = C_{d-1}^{\ell,k+\frac{1}{2}}$ and thus we have that

$$\mathcal{F}_d\phi_{\ell,k}(z) = \mathcal{F}_{d-1}\phi_{\ell,k+\frac{1}{2}}(z), \tag{2.7}$$

or equivalently,

$$\mathcal{F}_d\phi_{\frac{d+1}{2}+k,k}(z) = \mathcal{F}_{d-1}\phi_{\frac{d-1}{2}+k+1,k+\frac{1}{2}}(z). \tag{2.8}$$

We remark that (2.8) tells us that the $d-$dimensional Fourier transform (d odd) of the original Wendland function (designed for $\mathbb{R}^d$ with smoothness parameter k) coincides with the $d-1$ dimensional Fourier transform of the missing Wendland function, designed for $\mathbb{R}^{d-1}$ with smoothness parameter $k+\frac{1}{2}$. The following result summarises these discoveries.

Corollary 2.5. *Let k be a positive integer. If d is odd then the $d-$dimensional Fourier transform of the original Wendland function $\phi_{\ell,k}$ ($\ell = \frac{d+1}{2}+k$) is given by*

$$\mathcal{F}_d\phi_{\ell,k}(z) = \sqrt{\frac{2}{\pi}}\int_0^1 \phi_{\ell,\ell-1}(y)\cos(zy)dy. \tag{2.9}$$

Similarly, if d is even then the $d-$dimensional Fourier transform of the missing Wendland function $\phi_{\ell,k+\frac{1}{2}}$ ($\ell = \frac{d}{2}+k+1$) is given by

$$\mathcal{F}_d\phi_{\ell,k+\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}}\int_0^1 \phi_{\ell,\ell-1}(y)\cos(zy)dy. \tag{2.10}$$

Proof. The first formula can be derived by recursively applying (2.7) $d-1$ times to give

$$\mathcal{F}_d\phi_{\ell,k}(z) = \mathcal{F}_1\phi_{\ell,k+\frac{d-1}{2}}(z) = \mathcal{F}_1\phi_{\ell,\ell-1}(z).$$

Now, applying (1.2) with $d=1$ we find that

$$\mathcal{F}_1\phi_{\ell,\ell-1}(z) = \sqrt{z}\int_0^1 \phi_{\ell,\ell-1}(y)\sqrt{y}J_{-\frac{1}{2}}(zy)\,\mathrm{d}y = \sqrt{\frac{2}{\pi}}\int_0^1 \phi_{\ell,\ell-1}(y)\cos(zy)\,\mathrm{d}y,$$

where we have used the fact that

$$J_{-\frac{1}{2}}(t) = \sqrt{\frac{2}{\pi t}}\cos(t).$$

The same argument leads to the second formula. □

3 The functions $\phi_{\mu,k}$ where $k,\ell \in \mathbb{N}$

In this section we examine the generalised Wendland functions (1.3) in the special case where $\alpha = k$ and ℓ are positive integers. To initiate this investigation we present the following result.

Theorem 3.1. *Let d be a fixed space dimension and k be a positive integer. In addition let $\ell \geq (d+2k+1)/2$ be an integer. Then the function $\phi_{\ell,k}$ (1.3) is given by*

$$\phi_{\ell,k}(r) = \frac{1}{2^k k!}\sum_{j=0}^{k}\frac{\binom{k}{j}}{\binom{\ell+k+j}{\ell}}2^{k-j}r^{k-j}(1-r)^{\ell+k+j}, \qquad r\in[0,1]. \tag{3.1}$$

Proof. See [4] Theorem 4.1. □

We begin with an application of the binomial theorem to yield

$$\begin{aligned}\phi_{\ell,k}(r) &= \frac{1}{2^k k!}\sum_{j=0}^{k}\frac{\binom{k}{j}}{\binom{\ell+k+j}{\ell}}2^{k-j}\sum_{n=0}^{\ell+k+j}(-1)^n\binom{\ell+k+j}{n}r^{k+n-j} \\ &= \sum_{i=0}^{2k+\ell} c_i r^i,\end{aligned} \tag{3.2}$$

where, following some standard algebraic manipulation, the polynomial coefficient $(c_i)_{i=0}^{2k+\ell}$ are given by

$$
\begin{aligned}
c_i &= \frac{1}{2^k k!}\sum_{j=0}^{i}(-2)^j \frac{\binom{k}{j}}{\binom{\ell+2k-j}{\ell}}\binom{\ell+2k-j}{i-j} \\
&= \frac{\ell! k!}{2^k(\ell+2k-i)! i!}\sum_{j=0}^{i}(-2)^j \binom{2k-j}{k}\binom{i}{j} \\
&= \frac{\ell! 2k!}{2^k! k! (\ell+2k-i)! i!}\sum_{j=0}^{i}(-2)^j \frac{\binom{k}{j}}{\binom{2k}{j}}\binom{i}{j} \\
&= \frac{\ell! 2^k \Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}(\ell+2k-i)! i!}\sum_{j=0}^{i}(-2)^j \frac{\binom{k}{j}}{\binom{2k}{j}}\binom{i}{j},
\end{aligned}
\tag{3.3}
$$

where, in the final line we have employed the following formula from [1] Chapter 6, for evaluation of the Gamma function at the half-integers

$$\Gamma\left(k+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{4^k}\frac{(2k)!}{k!}, \qquad k = 0, 1, 2, \ldots \tag{3.4}$$

We can now employ the following identity taken from [11] (4.2.10.13)

$$\sum_{j=0}^{i}(-1)^j x^j \binom{i}{j}\frac{\binom{k}{j}}{\binom{2k}{j}} = \frac{\Gamma\left(k-i+\frac{1}{2}\right) i!}{\Gamma\left(k+\frac{1}{2}\right)}\left(\frac{-x}{4}\right)^i C_i^{(1/2+k-i)}\left(1-\frac{2}{x}\right),$$

where $C_i^{(\lambda)}$ denotes the Gegenbauer (or ultraspherical) polynomial of degree i and order λ (see [1] Chapter 22). Setting $x = 2$ in the above identity yields

$$\sum_{j=0}^{i}(-1)^j x^j \binom{i}{j}\frac{\binom{k}{j}}{\binom{2k}{j}} = \frac{\Gamma\left(k-i+\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)}(-1)^i \frac{i!}{2^i} C_i^{(1/2+k-i)}(0).$$

For a non-negative integer i we have (see [1] Section 22.4) that

$$C_i^{(\lambda)}(0) = \frac{2^i}{i!}\frac{\sqrt{\pi}\Gamma\left(\lambda+\frac{i}{2}\right)}{\Gamma(\lambda)\Gamma\left(-\frac{i-1}{2}\right)},$$

and so, using this identity, we can deduce that

$$\sum_{j=0}^{i}(-1)^j 2^j \frac{\binom{k}{j}}{\binom{2k}{j}}\binom{i}{j} = \frac{\Gamma\left(k-\frac{i-1}{2}\right)\sqrt{\pi}}{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(-\frac{i-1}{2}\right)} \tag{3.5}$$

,

and thus we have

$$\begin{aligned} c_i &= (-1)^i \frac{\ell! 2^k}{(\ell+2k-i)!i!}\frac{\Gamma\left(k-\frac{i-1}{2}\right)}{\Gamma\left(-\frac{i-1}{2}\right)} \\ &= (-2)^k \ell! \frac{(-1)^i \Gamma\left(\frac{i+1}{2}\right)}{\Gamma\left(\frac{i+1}{2}-k\right)(\ell+2k-i)!i!}, \end{aligned} \tag{3.6}$$

where, in the final line, we have used the reflection formula for the Gamma function ([1] 6.1.17)

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{3.7}$$

We are now in a position to deliver the following result.

Theorem 3.2. *Let d be a fixed space dimension and k be a positive integer. In addition let $\ell \geq (d+2k+1)/2$ be an integer. Then the function $\phi_{\ell,k}$ (3.1) is given by*

$$\phi_{\ell,k}(r) = (-2)^k \ell! \sum_{i=0}^{2k+\ell} \frac{(-1)^i \Gamma\left(\frac{i+1}{2}\right)}{\Gamma\left(\frac{i+1}{2}-k\right)(\ell+2k-i)!}\frac{r^i}{i!} \quad \text{for } r \in [0,1]. \tag{3.8}$$

Proof. This follows from (3.6). □

We close this section by collecting together some properties of the above family of polynomials. First of all we note that the first k odd coefficients of the polynomial are zero and so we may write $\phi_{\ell,k}(r)$ as

$$\begin{aligned} \phi_{\ell,k}(r) = (-2)^k \ell! \Bigg[&\sum_{i=0}^{k} \frac{\Gamma\left(i+\frac{1}{2}\right) r^{2i}}{\Gamma\left(i-k+\frac{1}{2}\right)(\ell+2k-2i)!(2i)!} \\ &- \sum_{i=0}^{\ell-1} \frac{(-1)^i \Gamma\left(k+1+\frac{i}{2}\right) r^{2k+1+i}}{\Gamma\left(1+\frac{i}{2}\right)(\ell-1-i)!(2k+1+i)!}\Bigg], \end{aligned} \tag{3.9}$$

or alternatively, as the sum of an even polynomial and a shorter odd polynomial

$$\phi_{\ell,k}(r) = (-2)^k \ell! \Bigg[\sum_{i=0}^{k+\lfloor\frac{\ell-1}{2}\rfloor} \frac{\Gamma\left(i+\frac{1}{2}\right) r^{2i}}{\Gamma\left(i-k+\frac{1}{2}\right)(\ell+2k-2i)!(2i)!} - \sum_{i=0}^{\lfloor\frac{\ell}{2}\rfloor-1} \frac{(k+i)! r^{2k+2i+1}}{i!(\ell-1-2i)!(2k+2i+1)!} \Bigg]. \tag{3.10}$$

This observation enables us to deduce that the first k odd derivatives of $\phi_{\ell,k}(r)$ vanish when evaluated at $r = 0$, i.e., we have

$$\phi_{\ell,k}^{(2p+1)}(0) = 0, \quad p = 0, 1, \ldots, k-1. \tag{3.11}$$

It is straightforward to deduce from (3.8) that the values of the remaining odd derivatives are given by

$$\phi_{\ell,k}^{(2k+1+2p)}(0) = \frac{(-1)^{k+1} 2^k \ell! (k+p)!}{(\ell-1-2p)! p!}, \quad p = 0, 1, \ldots, \left\lfloor \frac{\ell}{2} \right\rfloor - 1. \tag{3.12}$$

With (1.5), we can see that the first $(\ell + k - 1)$ derivatives of the function vanish at $r = 1$, i.e.,

$$\phi_{\ell,k}^{(n)}(1) = 0 \quad n = 0, 1, \ldots, \ell + k - 1. \tag{3.13}$$

4 Fourier transform of the original Wendland functions in odd dimensions

Throughout this section we shall assume that the space dimension d is odd. In this case we revisit the original Wendland functions $\phi_{\ell,k}$ with integer smoothness parameter k and $\ell = \frac{d+1}{2} + k$.

Before we embark on the calculation of the Fourier transform of the original Wendland functions, we briefly collect some important properties of the function $\phi_{\ell,\ell-1}$. Firstly, using Theorem 3.2 we know that it is a polynomial of degree $3\ell - 2$ on the unit interval. Specifically, (3.9) yields

$$\phi_{\ell,\ell-1}(y) = (-2)^{\ell-1} \ell! \Bigg[\sum_{i=0}^{\ell-1} \frac{\alpha_i y^{2i}}{(2i)!} - \sum_{i=0}^{\ell-1} \frac{\beta_i y^{2\ell-1+i}}{(2\ell-1+i)!} \Bigg], \tag{4.1}$$

where, for $i = 0, 1, \ldots, \ell - 1$, we have

$$\alpha_i = \frac{\Gamma\left(i + \frac{1}{2}\right)}{(3\ell - 2 - 2i)!\Gamma\left(i + \frac{1}{2} - (\ell - 1)\right)} \quad \text{and} \quad \beta_i = \frac{(-1)^i \Gamma\left(\ell + \frac{i}{2}\right)}{(\ell - 1 - i)!\Gamma\left(1 + \frac{i}{2}\right)}. \tag{4.2}$$

A combination of (3.11) and (3.12) allows us to deduce that

$$\phi_{\ell,\ell-1}^{(2p+1)}(0) = \begin{cases} 0 & \text{for } \; p = 0, 1, 2, \ldots, \ell - 2; \\ \frac{(-1)^\ell 2^{\ell-1} \ell! p!}{(p+1-\ell)!(3\ell-2p-3)!} & \text{for } \; p \geq \ell - 1. \end{cases} \tag{4.3}$$

Furthermore, in view of (3.13) we also have

$$\phi_{\ell,\ell-1}^{(n)}(1) = 0 \quad \text{for} \quad n = 0, 1, 2, \ldots, 2(\ell - 1). \tag{4.4}$$

Using (4.1) we can deduce that the value of the remaining ℓ derivatives at one are given by

$$\begin{aligned} \phi_{\ell,\ell-1}^{(2(\ell-1)+n)}(1) &= (-1)^\ell 2^{\ell-1} \ell! \sum_{i=n-1}^{\ell-1} \frac{\beta_i}{(i - (n-1))!} \\ &= (-1)^\ell 2^{\ell-1} \ell! \sum_{p=0}^{\ell-n} \frac{\beta_{p+n-1}}{p!} \\ &= (-2)^{\ell-1} \ell! (-1)^n \sum_{p=0}^{\ell-n} \frac{(-1)^p \Gamma\left(\ell - 1 + \frac{n+1+p}{2}\right)}{p!(\ell - n - p)!\Gamma\left(\frac{n+1+p}{2}\right)} \end{aligned} \tag{4.5}$$

for $n = 1, \ldots, \ell$.

Keeping these properties in mind we can now compute the Fourier transform as captured in the following theorem.

Theorem 4.1. *Let d be an odd space dimension, k a positive integer and let $\ell = (d + 2k + 1)/2$. The $d-$dimensional Fourier transform of the original Wendland function $\phi_{\ell,k}$, is given by $\mathcal{F}_d \phi_{\ell,k}(z) =$*

$$\frac{\sqrt{2/\pi}}{z^{d+2k+1}} \left[\cos(z) \sum_{j=0}^{\lfloor \frac{\ell-1}{2} \rfloor} \frac{a_{1,j}}{z^{2j}} + \sin(z) \sum_{j=0}^{\lfloor \frac{\ell}{2} \rfloor - 1} \frac{a_{2,j}}{z^{2j+1}} + \sum_{j=0}^{\lfloor \frac{\ell-1}{2} \rfloor} \frac{a_{3,j}}{z^{2j}} \right], \tag{4.6}$$

where

$$a_{1,j} = (-1)^j 2^{\ell-1}\ell! \sum_{p=0}^{\ell-2j-1} \frac{(-1)^p \Gamma\left(\ell+j+\frac{p}{2}\right)}{p!(\ell-2j-1-p)!\Gamma\left(j+1+\frac{p}{2}\right)}$$

$$a_{2,j} = (-1)^j 2^{\ell-1}\ell! \sum_{p=0}^{\ell-2j-2} \frac{(-1)^p \Gamma\left(\ell+j+\frac{p+1}{2}\right)}{p!(\ell-2j-2-p)!\Gamma\left(j+1+\frac{p+1}{2}\right)} \tag{4.7}$$

$$a_{3,j} = 2^{\ell-1}\ell! \frac{(-1)^j (j+\ell-1)!}{(\ell-1-2j)!j!}.$$

Proof. In view of Corollary 2.5 we have that $\mathcal{F}_d \phi_{\ell,k}(z) =$

$$\sqrt{\frac{2}{\pi}} \int_0^1 \phi_{\ell,\ell-1}(y)\cos(zy)\,\mathrm{d}y = \sqrt{\frac{2}{\pi}}\mathrm{Re}\left[\int_0^1 \phi_{\ell,\ell-1}(y)\exp(izy)\,\mathrm{d}y\right]$$

$$= \sqrt{\frac{2}{\pi}}\mathrm{Re}\left[\sum_{j=0}^{3\ell-2}(-1)^j \left[\frac{\phi^{(j)}_{\ell,\ell-1}(y)e^{izy}}{(iz)^{j+1}}\right]\Bigg|_{y=0}^{y=1}\right] \tag{4.8}$$

$$= \sqrt{\frac{2}{\pi}}\mathrm{Re}\left[\sum_{j=0}^{3\ell-2}(-1)^j \frac{\phi^{(j)}_{\ell,\ell-1}(1)e^{iz}}{(iz)^{j+1}} + \sum_{j=0}^{3\ell-2}(-1)^{j+1}\frac{\phi^{(j)}_{\ell,\ell-1}(0)}{(iz)^{j+1}}\right].$$

We shall examine the two sums appearing in the square brackets separately.

We begin with the first term:

$$\sum_{j=0}^{3\ell-2}(-1)^j\frac{\phi_{\ell,\ell-1}^{(j)}(1)e^{iz}}{(iz)^{j+1}} = -\sum_{j=0}^{3\ell-2} i^{j+1}\frac{\phi_{\ell,\ell-1}^{(j)}(1)e^{iz}}{z^{j+1}}$$

$$= -\sum_{j=2\ell-1}^{3\ell-2} i^{j+1}\frac{\phi_{\ell,\ell-1}^{(j)}(1)e^{iz}}{z^{j+1}} \quad \text{by (4.3)}$$

$$= \frac{(-1)^\ell}{z^{2\ell}}\left(\sum_{j=0}^{\ell-1}\frac{i^j\phi_{\ell,\ell-1}^{(2\ell-1+j)}(1)\cos(z)}{z^j} + \sum_{j=0}^{\ell-1}\frac{i^{j+1}\phi_{\ell,\ell-1}^{(2\ell-1+j)}(1)\sin(z)}{z^j}\right).$$

Noting that $2\ell = d+2k+1$ and taking the real part of this expression yields

$$\frac{1}{z^{d+2k+1}}\left(\cos(z)\sum_{j=0}^{\lfloor\frac{\ell-1}{2}\rfloor}\frac{(-1)^{\ell+j}\phi_{\ell,\ell-1}^{(2\ell-1+2j)}(1)}{z^{2j}} + \sin(z)\sum_{j=0}^{\lfloor\frac{\ell}{2}\rfloor-1}\frac{(-1)^{\ell+j+1}\phi_{\ell,\ell-1}^{(2\ell+2j)}(1)}{z^{2j+1}}\right)$$

$$= \frac{1}{z^{d+2k+1}}\left(\cos(z)\sum_{j=0}^{\lfloor\frac{\ell-1}{2}\rfloor}\frac{a_{1,j}}{z^{2j}} + \sin(z)\sum_{j=0}^{\lfloor\frac{\ell}{2}\rfloor-1}\frac{a_{2,j}}{z^{2j+1}}\right), \tag{4.9}$$

where, using (4.5) with $n = 2j+1$ and $n = 2j+2$, the coefficients $(a_{1,j})_{j\geq 0}$ and $(a_{2,j})_{j\geq 0}$ are as stated in the theorem.

We now move on to the second part of (4.8)

$$\sum_{j=0}^{3\ell-2}(-1)^{j+1}\frac{\phi_{\ell,\ell-1}^{(j)}(0)}{(iz)^{j+1}} = \sum_{j=0}^{3\ell-2}(-i)^{j+1}\frac{\phi_{\ell,\ell-1}^{(j)}(0)}{z^{j+1}}.$$

We note that the real part of this expression involves only the summands indexed by the odd integers. Furthermore, we know from (3.11) that the first $\ell-1$ odd derivatives of $\phi_{\ell,\ell-1}$ vanish at zero. Thus, in a similar fashion to the derivation of the first sum, we can deduce that

$$\operatorname{Re}\Big[\sum_{j=0}^{3\ell-2}(-i)^{j+1}\frac{\phi_{\ell,\ell-1}^{(j)}(0)}{z^{j+1}}\Big] = \frac{1}{z^{d+2k+1}}\sum_{j=0}^{\lfloor\frac{\ell-1}{2}\rfloor}\frac{(-1)^{\ell+j}\phi_{\ell,\ell-1}^{(2\ell-1+2j)}(0)}{z^{2j}} = \frac{1}{z^{d+2k+1}}\sum_{j=0}^{\lfloor\frac{\ell-1}{2}\rfloor}\frac{a_{3,j}}{z^{2j}} \tag{4.10}$$

where, applying (4.3) with $p=\ell-1+j$, the coefficients $(a_{3,j})_{j\geq 0}$ are as stated in the theorem. The result is established by substituting (4.9) and (4.10) into (4.8). □

5 Fourier transform of the original and missing Wendland functions in even dimensions

In this section, we derive closed form representations for the original and missing Wendland functions, for a given even spatial dimension d.

5.1 The original Wendland functions

Since the space dimension d is even, now we have $\ell = \frac{d}{2}+k+1$. Once more we can recursively apply (2.7) to deduce that

$$\mathcal{F}_d\phi_{\ell,k}(z) = \mathcal{F}_2\phi_{\ell,\ell-2}(z) = \mathcal{F}_1\phi_{\ell,\ell-\frac{3}{2}}(z).$$

We have developed closed form expressions for the generalised Wendland functions whose smoothness parameter is a positive integer, and consequently, for the current calculation, we shall focus on the 2−dimensional Fourier transform. Specifically, using (1.2) we shall compute:

$$\mathcal{F}_d\phi_{\ell,k}(z) = \int_0^1 \phi_{\ell,\ell-2}(y)yJ_0(zy)\,\mathrm{d}y. \tag{5.1}$$

As before we collect together the key properties concerning the function $\phi_{\ell,\ell-2}$. Starting with (3.8) we have its closed form representation

$$\phi_{\ell,\ell-2}(y) = \sum_{j=0}^{3\ell-4} c_j y^j, \quad \text{where} \quad c_j = \frac{(-2)^{\ell-2}\ell!(-1)^j\Gamma\left(\frac{j+1}{2}\right)}{\Gamma\left(\frac{j+1}{2}-(\ell-2)\right)(3\ell-4-j)!j!} \tag{5.2}$$

Now, from (3.13), we know that the first $2\ell - 3$ derivatives of $\phi_{\ell,\ell-2}$ vanish at one. Consequently, we can deduce from this that the polynomial coefficients $(c_j)_{j=0}^{3\ell-4}$ satisfy the following moment conditions:

$$\sum_{j=0}^{3\ell-4} c_j j^p = 0 \quad p = 0, 1, \ldots, 2(\ell-2), 2\ell-3, \tag{5.3}$$

and, in particular, we note that

$$\sum_{j=0}^{3\ell-4} c_j j^{2p} = 0, \quad p = 0, 1, \ldots, \ell-2.$$

With this preparation we can tackle the Fourier transform calculation

$$\mathcal{F}_d\phi_{\ell,k}(z) = \sum_{j=0}^{3\ell-4} c_j \int_0^1 y^{j+1} J_0(zy)\,\mathrm{d}y$$

Let

$$I_{j+1} = \int_0^1 y^{j+1} J_0(zy)\,\mathrm{d}y, \tag{5.4}$$

then, using [12] formula 1.8.1.5, we can deduce that

$$I_{j+1} = \frac{J_1(z)}{z} + \frac{j\,J_0(z)}{z^2} - \frac{j^2}{z^2} I_{j-1}. \tag{5.5}$$

Applying this, and taking account of the moment conditions (5.3), we can conclude that

$$\sum_{j=0}^{3\ell-4} c_j I_{j+1} = -\frac{1}{z^2}\sum_{j=0}^{3\ell-4} c_j j^2 I_{j-1}.$$

We can repeat this process $\ell - 1$ times to yield

$$\sum_{j=0}^{3\ell-4} c_j I_{j+1} = \frac{(-1)^{\ell-1}}{z^{2(\ell-1)}} \sum_{j=0}^{3\ell-4} c_j j^2 (j-2)^2 \cdots (j-2(\ell-1))^2 I_{j+1-2(\ell-1)}$$

$$= \frac{(-1)^{\ell-1}}{z^{2(\ell-1)}} \sum_{j=2(\ell-1)-1}^{3\ell-4} c_j j^2 (j-2)^2 \cdots (j-2(\ell-1))^2 I_{j+1-2(\ell-1)},$$

where the index shift of sum is valid since it is known that the first $\ell - 2$ odd coefficients of $\phi_{\ell,\ell-2}$ are zero. Using the identity

$$j^2(j-2)^2 \cdots (j-2(\ell-1))^2 = 2^{2(\ell-1)} \left(\frac{\Gamma\left(\frac{j}{2}+1\right)}{\Gamma\left(\frac{j}{2}+1-(\ell-1)\right)} \right)^2$$

we can, with a shift in the summation index, write

$$\mathcal{F}_d \phi_{\ell,k}(z) = \frac{(-1)^{\ell-1} 2^{2(\ell-1)}}{z^{2(\ell-1)}} \sum_{j=0}^{\ell-1} c_{2(\ell-1)-1+j} \left(\frac{\Gamma\left(\frac{j+1}{2}+\ell-1\right)}{\Gamma\left(\frac{j+1}{2}\right)} \right)^2 I_j.$$

We can now employ expression (5.2) for c_j, together with some algebraic manipulation, to show that the above sum simplifies to

$$\mathcal{F}_d \phi_{\ell,k}(z) = \frac{2^{\ell-1}\ell!}{z^{2(\ell-1)}} \sum_{j=0}^{\ell-1} \gamma_j I_j, \quad \text{where} \quad \gamma_j = \frac{(-1)^j \Gamma\left(\ell - 1 + \frac{j+1}{2}\right)}{(\ell-1-j)!\,\Gamma\left(\frac{j+1}{2}\right) j!} \tag{5.6}$$

Now, for $j \geq 2$ we can rewrite (5.5) as

$$I_j = \frac{J_1(z)}{z} + \frac{(j-1)J_0(z)}{z^2} - \frac{(j-1)^2}{z^2} I_{j-2}. \tag{5.7}$$

Furthermore, appealing to [5] (formulae 6.561.1. and 6.561.5), we have that

$$I_0 = \Lambda(z) + J_0(z) \quad \text{and} \quad I_1 = \frac{J_1(z)}{z}, \tag{5.8}$$

where the function $\Lambda(z)$ is defined by

$$\Lambda(z) := \frac{\pi}{2}\left(J_1(z)H_0(z) - J_0(z)H_1(z)\right) \tag{5.9}$$

where $H_\nu(z)$ denotes the Struve function of order ν; see [1] Chapter 12.

We can now employ (5.8) and (5.7) together to deduce that

$$\begin{aligned}\sum_{j=0}^{\ell-1}\gamma_j I_j =& \Lambda(z)\gamma_0 + J_1(z)\left[\frac{\sum_{j=0}^{\ell-2}\gamma_{j+1}}{z}\right] \\ &+ J_0(z)\left[\gamma_0 + \frac{\sum_{j=0}^{\ell-3}\gamma_{j+2}(j+1)}{z^2}\right] - \frac{1}{z^2}\sum_{j=0}^{\ell-3}(j+1)^2\gamma_{j+2}I_j.\end{aligned}$$

Repeating this process again leads to

$$\begin{aligned}\sum_{j=0}^{\ell-1}\gamma_j I_j =& \Lambda(z)\left(\gamma_0 - \frac{\gamma_2 1^2}{z^2}\right) \\ &+ J_1(z)\left[\frac{\sum_{j=0}^{\ell-2}\gamma_{j+1}}{z} - \frac{\sum_{j=0}^{\ell-4}\gamma_{j+3}(j+2)^2}{z^3}\right] \\ &+ J_0(z)\left[\gamma_0 + \frac{\sum_{j=1}^{\ell-3}\gamma_{j+2}(j+1)}{z^2} - \frac{\sum_{j=0}^{\ell-5}\gamma_{j+4}(j+1)(j+3)^2}{z^4}\right] \\ &+ \frac{1}{z^4}\sum_{j=0}^{\ell-5}\gamma_{j+4}(j+1)^2(j+3)^2 I_j.\end{aligned}$$

Clearly, we can continue applying this recursion to deliver a closed formula for the even dimensional Fourier transform. In order to express this neatly we use the above development to fix formulae for the coefficients involved. Firstly, using (5.6) and (3.4) we set

$$b_{1,p} = (-1)^p \gamma_{2p} 2^{2p} \left(\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \right)^2 = \frac{(-1)^p \Gamma\left(\ell - 1 + p + \frac{1}{2}\right)}{(\ell-1-2p)!p!\sqrt{\pi}}. \tag{5.10}$$

Next, we set

$$\begin{aligned} b_{2,p} &= (-1)^p 2^{2p} \sum_{j=0}^{\ell-2(p+1)} \gamma_{2p+1+j} \left(\frac{\Gamma\left(p+1+\frac{j}{2}\right)}{\Gamma\left(1+\frac{j}{2}\right)} \right)^2 \\ &= (-1)^p 2^{2p} \sum_{j=0}^{\ell-2(p+1)} \frac{(-1)^{j+1}\Gamma\left(p+\ell+\frac{j}{2}\right)\Gamma\left(p+1+\frac{j}{2}\right)}{(\ell-2(p+1)-j)!(2p+1+j)!\Gamma\left(1+\frac{j}{2}\right)^2}, \end{aligned} \tag{5.11}$$

and finally

$$b_{3,p} = \begin{cases} \gamma_0 & \text{for } \; p=0; \\ (-1)^{p+1} 2^{2p-1} \sum_{j=1}^{\ell-2p-1} \gamma_{2p+j} \frac{\Gamma\left(p+\frac{j+1}{2}\right)^2}{\Gamma\left(\frac{j+1}{2}\right)\Gamma\left(\frac{j+3}{2}\right)} & \text{for } \; p \geq 1. \end{cases}$$

$$= \begin{cases} \gamma_0 & \text{for } \; p=0; \\ (-1)^{p+1} 2^{2p-1} \sum_{j=1}^{\ell-2p-1} \frac{(-1)^j \Gamma\left(\ell-1+p+\frac{j+1}{2}\right)\Gamma\left(p+\frac{j+1}{2}\right)}{(\ell-1-2p-j)!(2p+j)!\Gamma\left(\frac{j+1}{2}\right)\Gamma\left(\frac{j+3}{2}\right)} & \text{for } \; p \geq 1. \end{cases} \tag{5.12}$$

With the coefficients prepared we can conclude the following result.

Theorem 5.1. *Let d be an even space dimension, k a positive integer and let $\ell = \frac{d}{2} + k + 1$. The $d-$dimensional Fourier transform of the original Wendland function $\phi_{\ell,k}$, is given by $\mathcal{F}_d \phi_{\ell,k}(z) =$*

$$\frac{\ell! 2^{\ell-1}}{z^{d+2k}} \left[\Lambda(z) \sum_{p=0}^{\lfloor \frac{\ell-1}{2} \rfloor} \frac{b_{1,p}}{z^{2p}} + J_1(z) \sum_{p=0}^{\lfloor \frac{\ell}{2} \rfloor - 1} \frac{b_{2,p}}{z^{2p+1}} + J_0(z) \sum_{p=0}^{\lfloor \frac{\ell}{2} \rfloor - 1} \frac{b_{3,p}}{z^{2p}} \right] \tag{5.13}$$

where the coefficients $b_{1,j}, b_{2,j}$ and $b_{3,j}$ are given by (5.10), (5.11) and (5.12) respectively.

We note that it is not immediately obvious from (5.13) that we achieve the asymptotic decay rate predicted by Theorem 2.3, namely that

$$\mathcal{F}_d\phi_{\ell,k}(z) = O\left(\frac{1}{z^{d+2k+1}}\right). \tag{5.14}$$

In view of this we close the paper by directly investigating the asymptotic behaviour of (5.13). To begin with we notice that a glance at (5.10) and (5.12) shows that $b_{3,0} = \gamma_0 = b_{1,0}$ and so, isolating the first terms ($p = 0$) which contribute from the sums in (5.13) we can write $\mathcal{F}_d\phi_{\ell,k}(z) =$

$$\frac{\ell! 2^{\ell-1}}{z^{d+2k}}\left[\gamma_0\left(\Lambda(z) + J_0(z)\right) + \frac{b_{2,0}J_1(z)}{z} + (\text{rapidly decaying terms})\right]. \tag{5.15}$$

Using the asymptotic expansions of the Bessel function [1] (formula 9.2.1) and Struve functions [2] (formulae 3.62 and 3.63), we see that as $z \to \infty$

$$\begin{aligned}
\Lambda(z) &= -\sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{4}\right) + O(z^{-1}) \\
J_0(z) &= \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{4}\right) + O(z^{-1}) \\
J_1(z) &= \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{3\pi}{4}\right) + O(z^{-1})
\end{aligned}$$

and consequently, as $z \to \infty$, we have

$$\Lambda(z) + J_0(z) = O\left(\frac{1}{z}\right) \quad \text{and} \quad \frac{J_1(z)}{z} = O\left(\frac{1}{z^{3/2}}\right).$$

From this, we can verify that the Fourier transform decays at the expected rate (5.14).

5.2 The missing Wendland functions

Since we now are working with the missing Wendland functions in an even dimension d, we seek a closed form for $\mathcal{F}_d\phi_{\ell,k+\frac{1}{2}}$, where k is a positive integer and $\ell = d/2 + k + 1$. With (2.7), we can see that

$$\begin{aligned}
\mathcal{F}_d\phi_{\frac{d}{2}+k+1,k+\frac{1}{2}} &= \mathcal{F}_{d-1}\phi_{\frac{d}{2}+k+1,k+1} \\
&= \mathcal{F}_{d-1}\phi_{\frac{d-1}{2}+k+1+\frac{1}{2},k+1},
\end{aligned}$$

which is just the $d-1$-dimensional Fourier transform of the original Wendland function with smoothness parameter $k+1$ (since k is an integer). Since $d-1$ is odd, the closed form representation for this is given in Theorem 4.1, which gives the following result.

Theorem 5.2. *Let d be an even space dimension, k a positive integer and let $\ell = d/2+k+1$. The $d-$dimensional Fourier transform of the missing Wendland function $\phi_{\ell,k+\frac{1}{2}}$, is given by $\mathcal{F}_d\phi_{\ell,k+\frac{1}{2}}(z) =$*

$$\frac{\sqrt{2/\pi}}{z^{d+2k+2}}\left[\cos(z)\sum_{j=0}^{\lfloor\frac{\ell-1}{2}\rfloor}\frac{a_{1,j}}{z^{2j}}+\sin(z)\sum_{j=0}^{\lfloor\frac{\ell}{2}\rfloor-1}\frac{a_{2,j}}{z^{2j+1}}+\sum_{j=0}^{\lfloor\frac{\ell-1}{2}\rfloor}\frac{a_{3,j}}{z^{2j}}\right], \tag{5.16}$$

where $a_{1,j}, a_{2,j}$ and $a_{3,j}$ are given by (4.7).

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